Series expansions for the Potts model: high-field expansions

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Abstract. High-field expansions are obtained for the three-state Potts model. These expansions are used to investigate the critical isotherm of the three-state system on a square lattice, and to determine the general properties of the transition in face-centred cubic systems.

1. Introduction

After a period of comparative neglect there has recently been a renewed interest in the three-state Potts model. Straley and Fisher (1973) have undertaken an extensive analysis of the square lattice system and have reviewed earlier work on the model. Alexander and Yuval (1973) have recently discussed series expansions, while Golner (1973) has used renormalization group techniques to investigate the model.

One of the most interesting questions is the order of the transition. Baxter (1973) has indicated that the general q-state non-planar Potts model on a square lattice should have a first-order transition for $q > 4$. Ditzian and Oitmaa (1974) have suggested that the three-state model on a face-centred cubic lattice has a first-order transition. Golner finds that the transition is first-order in three dimensions.

It seems quite probable that the field representations do not properly represent the Potts model. The Baxter result of a continuous transition for three-state systems appears to be contradicted by the results of Golner in two dimensions, and also by the work of Amit and Shcherbakov (1974) who treated a continuous Potts model by the Callan–Symanzik technique. The results in the present work indicate that in three dimensions there is a similar discrepancy between the results of discrete models and results for field models.

The work presented here gives high-field expansions for the three-state model. Five high-field polynomials are obtained for the square, simple cubic, body-centred cubic, triangular and face-centred cubic lattices. The FCC series are used to study the transition. The five polynomials give a sufficient number of terms in the temperature grouping to give some hope of reliable results from the analysis. On lattices other than FCC a smaller number of temperature grouping terms are obtained. For these lattices additional high-field polynomials could be obtained by using the code method (Sykes *et al* 1965). This has been done for the square lattice and an estimate of $\delta = 15.0 \pm 0.5$ has been obtained. The additional terms in the temperature grouping do not however extend the series given by Straley and Fisher and so we evaluated the expressions only at T_c.

The behaviour on the FCC lattice is still not entirely clear. If the transition is interpreted as continuous then the exponents have the same general pattern as the exponents

on the square lattice. The critical point estimates are however widely scattered and predominantly above the critical point estimated from high temperature expansions. Estimates of the critical energy on each side of the transition seem to indicate that the transition is indeed continuous.

The layout of the paper is as follows. Section 2 discusses high-field expansions, § 3 considers the critical isotherm on the square lattice. Section 4 considers the FCC lattice. The high-field polynomials are given in the appendix, as are the 'code' expressions for the square lattice.

2. High-field expansions

The series that will be derived are expansions about a fully aligned state. All possible perturbations of up to five sites are considered and each perturbation is given the appropriate Boltzmann weighting $u^p \mu_1^n \mu_2^m$ where n, m sites are perturbed from state 3 into states 1 and 2, respectively, and p is the number of neighbour pairs that are not aligned.

$$u = \exp(-\beta J) \tag{1}$$

$$\mu_1 = \exp(-\beta h_1) \tag{2}$$

$$\mu_2 = \exp(-\beta h_2) \tag{3}$$

where h_1, h_2 are fields and J is the energy difference between aligned and non-aligned neighbour pairs. Each perturbation can be mapped onto a strong graph on the lattice, and if one considers graphs whose vertices are labelled 1 or 2 then there is a one-to-one correspondence. The numbers n, m are found by counting vertices and the number p is found by using a 'linkage rule' of the type described by Sykes and Gaunt (1973). Since p depends only on the topology of the graph and not on how it is embedded, it is possible to obtain expansions by considering all possible 'decorations' of the graphs given by Domb (1960). Straley and Fisher (1973) and Alexander and Yuval (1973) have effectively described the linkage rule in terms of weak graphs on the dual lattice. This is the basis of the duality transformation that can be used to obtain the transition temperature on the square lattice but it is not applicable to series in three dimensions.

The configurational free energy is expanded as

$$\ln \Lambda = \sum_{knm} l_{knm} u^k \mu_1^n \mu_2^m = \sum_{nm} L_{nm}(u) \mu_1^n \mu_2^m \tag{4}$$

where the L_{nm} are finite polynomials in u with the properties

$$L_{nm}(u) = L_{mn}(u) \tag{5}$$

$$L_{n0}(u) = L_n(u^2) \tag{6}$$

where the L_n are the high-field polynomials for the spin $\frac{1}{2}$ Ising model and are given by Sykes *et al* (1965). Other L_{nm} are given in the appendix.

Another way of grouping the expansions on loose-packed lattices is to distinguish between fields on each of two sublattices A and B where all pairs of neighbours lie one on each sublattice.

Writing expansions as

$$\ln \Lambda = \sum_{nm} \bar{L}_{nm}(u) \mu_A^n \mu_B^m \tag{7}$$

it is possible to obtain expressions for

$$F_n(u, \mu) = \sum_m \bar{L}_{nm}(u)\mu^m.$$ (8)

The F_n are obtained from strong graphs on a shadow lattice and the graphical information is expressed as codes. The codes represent contributions to F_n by means of the notation

$$c(\alpha, \beta, \gamma, \delta, \ldots) \Rightarrow cf_1^{-\alpha} f_2^{\beta} f_3^{\gamma} \ldots.$$ (9)

For a q-state model on a lattice of coordination number z, appropriate f_i are

$$f_1 = 1 + (q-1)\mu u^z$$ (10)

$$f_2 = 1 + \mu u^{z-2} + (q-2)\mu u^{z-1}$$ (11)

$$f_3 = 1 + \mu u^{z-4} + (q-2)\mu u^{z-2}$$ (12)

$$f_4 = 1 + 2\mu u^{z-3} + (q-3)\mu u^{z-2}$$ (13)

$$f_5 = 1 + \mu u^{z-6} + (q-2)\mu u^{z-3}$$ (14)

$$f_6 = 1 + \mu u^{z-5} + \mu u^{z-4} + (q-3)\mu u^{z-3}$$ (15)

$$f_7 = 1 + 3\mu u^{z-4} + (q-4)\mu u^{z-3}$$ (16)

$$f_8 = 1 + \mu u^{z-8} + (q-2)\mu u^{z-4}$$ (17)

$$f_9 = 1 + \mu u^{z-7} + \mu u^{z-5} + (q-3)\mu^{z-4}$$ (18)

$$f_{10} = 1 + 2\mu u^{z-6} + (q-3)\mu u^{z-4}$$ (19)

$$f_{11} = 1 + \mu u^{z-6} + 2\mu u^{z-5} + (q-4)\mu u^{z-4}$$ (20)

$$f_{12} = 1 + 4\mu u^{z-5} + (q-5)\mu u^{z-4}.$$ (21)

For $q = 3$ code expansions of F_0 to F_4 for the square lattice are given in the appendix.

3. The critical isotherm of the square lattice

As remarked in the introduction, the codes have not been expanded in full to obtain $\bar{L}_{nm}(u)$ but have been evaluated at $u_c = \frac{1}{2}(\sqrt{3} - 1)$ to give $\bar{L}_{nm}(u_c)$ and thence give the series

$$\ln \Lambda(\mu, u_c) = \sum_n \mu^n d_n.$$ (22)

All these code expansions are restricted to the case $\mu_1 = \mu_2 = \mu$. More general codes could be obtained and may be useful in determining the full field dependence of the transition temperature.

The spontaneous order is

$$\Psi = 1 - \frac{3}{2}\mu \frac{\partial}{\partial \mu} \ln \Lambda.$$ (23)

If one has

$$\Psi \sim (1 - \mu)^{1/\delta}$$ (24)

then

$$-\mu \frac{d}{d\mu} \ln \Psi = \sum e_n \mu^n \tag{25}$$

with

$$e_n \sim 1/\delta. \tag{26}$$

This method of obtaining the e_n to give estimates of $1/\delta$ was found to be very successful in the Ising model (Gaunt and Sykes 1972). The $(e_n)^{-1}$ estimates are given in table 1 and extrapolate to give $\delta = 15 \cdot 0 \pm 0 \cdot 5$ for the Potts model.

Table 1. Estimates for the exponent δ for the three-state Potts model on a square lattice. Obtained by taking $-\mu(d/d\mu) \ln \Psi(u_c, \mu)$. See equations (22), (24) to (26).

n	1	2	3	4	5	6	7	8	9
$\frac{1}{2} n d_n$	0·05385	0·02975	0·01763	0·01427	0·01092	0·00941	0·00781	0·00693	0·00609
$(e_n)^{-1}$	18·57	16·02	17·28	15·87	16·24	15·59	15·86	15·54	15·56

The series estimates obtained by Straley and Fisher (1973) were

$$\alpha' = 0 \cdot 05 \pm 0 \cdot 1 \tag{27}$$

$$\beta = 0 \cdot 103 \pm 0 \cdot 01 \tag{28}$$

$$\gamma' = 1 \cdot 5 \pm 0 \cdot 2 \tag{29}$$

$$\gamma'_\perp = 1 \cdot 1 \pm 0 \cdot 05 \tag{30}$$

but if one assumes the upper limits as bounds the Rushbrooke inequality implies

$$\alpha' \geqslant 0 \cdot 07 \tag{31}$$

$$\beta \geqslant 0 \cdot 07 \tag{32}$$

$$\gamma' \geqslant 1 \cdot 62. \tag{33}$$

The estimate $\delta = 15 \pm 0 \cdot 5$ together with the Griffith inequality $\alpha' + \beta(1 + \delta) \geqslant 2$ gives more stringent restrictions:

$$\alpha' \geqslant 0 \cdot 13 \tag{34}$$

$$\beta \geqslant 0 \cdot 112 \tag{35}$$

while $\delta = 15$ requires that at least one of α', β lies outside the range estimated by Straley and Fisher, but it is possible for them to be very near the range given so long as both α', β lie near the upper limits of the estimates. In view of the value $\delta = 15$ occurring in a number of short-range two-dimensional systems it seems plausible that $\delta = 15$ for the Potts model.

4. The face-centred cubic lattice

Unlike the square lattice the critical point is not known exactly for the FCC lattice so it is difficult to accurately determine the exponents, or even accurately determine the

transition temperature. Indeed, there is even doubt concerning the order of the transition. Some of the results presented here seem to indicate a continuous transition, but the renormalization group arguments of Golner (1973) predict a first-order transition and Ditzian and Oitmaa (1974) suggest that the transition is first order.

In this section we consider the series

$$\ln Z = \tfrac{1}{2}Nz \ln(x+2) - \tfrac{1}{2}\beta JNz + \sum a_n w^n \tag{36}$$

$$x = \exp(\beta J) \tag{37}$$

$$w = \frac{x-1}{x+2} \tag{38}$$

whence

$$U = -\frac{\partial}{\partial \beta} \ln Z = -\left(\frac{Jx}{x+2}\right)\tfrac{1}{2}Nz + \tfrac{1}{2}JNz - xJ\frac{3}{(x+2)^2}\sum a_n w^{n-1} n. \tag{39}$$

This is based on taking the fully aligned state as the zero of energy. z is the coordination number. The high-temperature expansion coefficients are obtained using techniques described by Domb (1974). Further details of the expansions for high temperatures will be given elsewhere.

The low-temperature series considered are based on $\ln \Lambda = \Sigma \, l_{knm} u^k \mu_1^n \mu_2^m$ and are:

spontaneous order

$$\Psi = 1 - \tfrac{3}{2} \sum_{k,n,m} (n+m) u^k l_{knm} \tag{40}$$

specific heat

$$C = \beta J^2/T \sum_{knm} l_{knm} k^2 u^k \tag{41}$$

energy

$$U = J \sum_{knm} l_{knm} k u^k \tag{42}$$

parallel susceptibility

$$\chi_{\parallel} = \beta \sum_{knm} l_{knm}(n+m)^2 u^k \tag{43}$$

perpendicular susceptibility

$$\chi_{\perp} = \beta \sum_{knm} l_{knm}(n-m)^2 u^k. \tag{44}$$

For critical point estimates, the work of Ditzian and Oitmaa (1974) gives

$$u_c = 0.769 \pm 0.003$$

on the basis of a six-term high-temperature susceptibility series. They suggest that this may not actually represent the transition temperature as the transition may be first order.

By considering the statistics of self-avoiding walks, Domb (1974) obtains the estimate

$$w_c^{-1} = 11.15 \pm 0.2 \tag{45}$$

or

$$u_c = 0.773 \pm 0.003. \tag{46}$$

This agreement of high-temperature estimates would still be quite possible even if the transition was first order.

The most regular low-temperature critical-point estimates come from Padé approximants to the logarithmic derivative of Ψ. It was first necessary to locate a pair of singularities at $0.541 \pm 0.002 \pm i(0.442 \pm 0.002)$. Almost all approximants had one or more singularities in these general regions. Over 50% of the approximants had a single conjugate pair of singularities within the range given above. This group seemed to give the most regular estimates of the physical singularity, and on the basis of results from this group we obtain

$$u_c = 0.770 \pm 0.002. \tag{47}$$

This result would tend to strongly suggest a continuous transition but the results for C, $\chi_\parallel$ and $\chi_\perp$ gave many estimates of $u_c > 0.77$.

Briefly summarizing the results: 30% of approximants to $(d/du) \ln \chi_\parallel$ had a pair of roots in the regions

$$0.531 \pm 0.002 \pm i(0.455 \pm 0.002).$$

This set of approximants leads to

$$u_c = 0.772 \pm 0.005.$$

The logarithmic derivative to the specific heat appears to have a pair of roots somewhere near 0.533 ± 0.452. No consistent pattern of u_c estimates could be obtained.

The perpendicular susceptibility also gave no very coherent pattern of u_c estimates although when a root near 0.77 did occur it was generally somewhat above 0.77. There appears to be a pair of complex roots at $0.542 \pm 0.002 \pm i(0.468 \pm 0.002)$.

Both the amount of scatter of u_c estimates and the 'overshoot' of the estimates from $\chi_\perp$ argue against a continuous transition.

Additional evidence concerning the nature of the transition is obtained by extrapolating the energy series (39) and (42) to various transition temperatures.

For $u_c = 0.770$ the low temperature series (42) gives on the basis of Padé approximants to U

$$U_c/J = 2.48 \pm 0.02. \tag{48}$$

The high-temperature series reduces to

$$U_c/J = 6.0 - 2.36 - 0.98 \sum_n a_n w_c^{n-1} n \tag{49}$$

with

$$\sum_n a_n w_c^{n-1} n = 48w_c^2 + 264w_c^3 + 2040w_c^4 + 15\,528w_c^5 + 125\,412w_c^6 + 652\,368w_c^7 + \dots. \tag{50}$$

Because, as pointed out by Straley and Fisher (1973), Padé approximants cannot really represent the energy, the high-temperature series was analysed by constructing the partial sums:

$$U_m = \sum_{n=3}^{m} a_n n w_c^{n-1}. \tag{51}$$

It was found that the U_m varied linearly with $1/m$ and extrapolated to 1.15 ± 0.05. Thus assuming $u_c = 0.77$ gives agreement between the critical energies, the high-temperature estimate being 2.51 ± 0.05. Since the low-temperature series did not have the same

regularity as the high-temperature series and had to be analysed by Padé approximants, it is probable that a larger 'error range' should be assigned to estimate (48). Although the accuracy of the energy estimates is not outstanding they do appear to indicate a continuous transition.

Assuming that the transition is continuous we can now return to the Padé approximants to Ψ, $\chi_{\|}$, $\chi_{\perp}$ and C and investigate the critical exponents. This was done in the normal manner of constructing Padé approximants to $(x_c - x)(\mathrm{d}/\mathrm{d}x)\ln f(x)$ for the various functions $f(x)$. In actual fact various powers of x were removed so that the series began with x^0. The exponent estimates obtained were

$$\alpha' = 0.37 \pm 0.10 \tag{52}$$

$$\beta = 0.184 \pm 0.004 \tag{53}$$

$$\gamma' = 1.0 \pm 0.15 \tag{54}$$

$$\gamma'_{\perp} = 0.6 \qquad \text{(possibly).} \tag{55}$$

A similar value for β has been obtained independently (R V Ditzian, private communication).

The estimates for α', γ', $\gamma'_{\perp}$ were quite irregular. The $\gamma'_{\perp} \simeq 0.6$ was a common estimate but no meaningful error bounds can be assigned to this result since only half the approximants gave values anywhere near 0.6. Attempts to estimate δ found, not surprisingly, that the five terms available were not sufficient to give consistent estimates. α', β, γ' have the same general behaviour as found by Straley and Fisher (1973) for the square lattice model; α' is in each case somewhat greater than the Ising model value (remembering that the δ estimate in § 3 requires a fairly large α' in two dimensions). The β estimates were in each case somewhat below the Ising model values, as were the γ' estimates. The range of values obtained is consistent with the Rushbrooke inequality holding as an equality, that is, that scaling theory holds. It is not clear how to incorporate the exponent $\gamma'_{\perp}$ into a scaling scheme.

If the exponents for the Potts model are in fact simple fractions, then it would be of interest to see how these fractions vary with the number of states. It is however reasonable to suppose that the fractions will in general be not quite as 'simple' as those for the Ising model, and so it seems pointless to guess what fractions might be represented by the estimates (52) to (55).

5. Conclusions

While the work in the preceding sections has given a considerable amount of new information concerning the Potts model, general concepts of the behaviour of the model are lacking. In particular, the way in which the order of the transition changes with the dimensionality d, and the number of states q remains unclear. Ditzian and Oitmaa (1974) base their discussion on the position of the tricritical point but Straley and Fisher (1973) have cast doubt on the mean-field approximation type of tricritical point. It has been pointed out (D S Gaunt, private communication) that the Baxter result is consistent with a first-order transition for $d + q > 4$ (for $d \leqslant 4$). Implicit in both these arguments is the assumption that the mean-field approximation is correct in predicting a first-order transition for $d = 4$, $q = 3$, but this prediction has not really been verified.

As remarked in earlier sections, there are a number of ways of obtaining extended series. Domb (1974) has shown how to obtain high-temperature free energy expansions. For low temperatures, high-field expansions obtained by the code method are available. The case of the triangular lattice is of particular interest. One divides the lattice into three sublattices. When combining the expanded generating functions a field with components staggered between the three sublattices can be included. Such a field can be the ordering field for a three-state system in which non-parallel alignments are favoured. On the triangular lattice such an 'antiferromagnetic' system has a ground state that is unique (to within permutations) while on other lattices the 'antiferromagnetic' ground state is highly degenerate. Work on these and other aspects of the Potts model is currently in progress.

Acknowledgments

Appendix

A.1. High-field polynomials for three-state Potts model

Square lattice

$$L_{11} = 4u^7 - 5u^8$$

$$L_{21} = 12u^9 - 10u^{10} - 32u^{11} + 31u^{12}$$

$$L_{31} = 4u^{10} + 40u^{11} - 34u^{12} - 212u^{13} + 174u^{14} + 236u^{15} - 209u^{16}$$

$$L_{22} = 4u^{10} + 28u^{11} + 8u^{12} - 220u^{13} + 20u^{14} + 472u^{15} - 313\tfrac{1}{2}u^{16}$$

$$L_{41} = 8u^{11} + 12u^{12} + 140u^{13} - 154u^{14} - 1172u^{15} + 919u^{16}$$
$$+ 2608u^{17} - 2092u^{18} - 1744u^{19} + 1476u^{20}$$

$$L_{32} = 24u^{12} + 128u^{13} - 52u^{14} - 1000u^{15} - 510u^{16} + 4000u^{17}$$
$$- 308u^{18} - 5232u^{19} + 2952u^{20}.$$

Simple cubic

$$L_{11} = 6u^{11} - 7u^{12}$$

$$L_{21} = 30u^{15} - 21u^{16} - 72u^{17} + 64u^{18}$$

$$L_{31} = 12u^{18} + 186u^{19} - 117u^{20} - 808u^{21} + 567u^{22} + 810u^{23} - 651u^{24}$$

$$L_{22} = 12u^{18} + 126u^{19} + 81u^{20} - 846u^{21} - 18u^{22} + 1620u^{23} - 976\tfrac{1}{2}u^{24}$$

$$L_{41} = 48u^{21} + 84u^{22} + 1254u^{23} - 917u^{24} - 8040u^{25} + 5058u^{26}$$
$$+ 15\,310u^{27} - 10\,611u^{28} - 9216u^{29} + 7031u^{30}$$

$$L_{32} = 144u^{22} + 978u^{23} + 357u^{24} - 6792u^{25} - 4920u^{26} + 23\,338u^{27}$$
$$+ 483u^{28} - 27\,648u^{29} + 14\,062u^{30}.$$

Body-centred cubic

$$L_{11} = 8u^{15} - 9u^{16}$$

$$L_{21} = 56u^{21} - 36u^{22} - 128u^{23} + 109u^{24}$$

$$L_{31} = 48u^{26} + 464u^{27} - 300u^{28} - 1944u^{29} + 1300u^{30} + 1896u^{31} - 1465u^{32}$$

$$L_{22} = 48u^{26} + 296u^{27} + 252u^{28} - 2088u^{29} - 104u^{30} + 3792u^{31} - 2197\tfrac{1}{2}u^{32}$$

$$L_{41} = 36u^{30} + 240u^{31} + 348u^{32} + 4152u^{33} - 3492u^{34} - 26\,184u^{35}$$
$$+ 16\,306u^{36} + 48\,336u^{37} - 32\,244u^{38} - 28\,368u^{39} + 20\,871u^{40}$$

$$L_{32} = 72u^{31} + 684u^{32} + 3392u^{33} + 420u^{34} - 21\,784u^{35} - 17\,000u^{36}$$
$$+ 74\,256u^{37} + 3324u^{38} - 85\,104u^{39} + 41\,742u^{40}.$$

Triangular

$$L_{11} = 6u^{11} - 7u^{12}$$

$$L_{21} = 6u^{14} + 18u^{15} - 21u^{16} - 60u^{17} + 58u^{18}$$

$$L_{31} = 6u^{16} + 18u^{17} + 72u^{19} - 135u^{20} - 478u^{21} + 459u^{22} + 576u^{23} - 519u^{24}$$

$$L_{22} = 12u^{17} + 30u^{18} + 102u^{19} - 201u^{20} - 414u^{21} + 96u^{22} + 1152u^{23} - 778\tfrac{1}{2}u^{24}$$

$$L_{41} = 12u^{18} + 30u^{19} + 126u^{21} - 123u^{22} - 156u^{23} - 347u^{24} - 2994u^{25}$$
$$+ 3348u^{26} + 7942u^{27} - 7101u^{28} - 5592u^{29} + 4856u^{30}$$

$$L_{32} = 12u^{19} + 42u^{20} + 126u^{21} + 36u^{22} + 42u^{23} - 795u^{24} - 3858u^{25}$$
$$+ 2094u^{26} + 11\,290u^{27} - 1923u^{28} - 16\,776u^{29} + 9712u^{30}.$$

Face-centred cubic

$$L_{11} = 12u^{23} - 13u^{24}$$

$$L_{21} = 24u^{32} + 84u^{33} - 78u^{34} - 240u^{35} + 211u^{36}$$

$$L_{31} = 8u^{39} + 48u^{40} + 168u^{41} + 52u^{42} + 816u^{43} - 1158u^{44}$$
$$- 4324u^{45} + 3522u^{46} + 4644u^{47} - 3777u^{48}$$

$$L_{22} = 12u^{40} + 96u^{41} + 300u^{42} + 1044u^{43} - 1428u^{44} - 3924u^{45}$$
$$+ 276u^{46} + 9288u^{47} - 5665\tfrac{1}{2}u^{48}$$

$$L_{41} = 24u^{46} + 96u^{47} + 142u^{48} + 624u^{49} + 312u^{50} + 2896u^{51}$$
$$- 1794u^{52} - 372u^{53} - 8734u^{54} - 65\,124u^{55} + 58\,947u^{56}$$
$$+ 145\,144u^{57} - 112\,092u^{58} - 91\,584u^{59} + 71\,516u^{60}$$

$$L_{32} = 96u^{48} + 336u^{49} + 876u^{50} + 2584u^{51} + 1872u^{52} + 4056u^{53} - 13\,672u^{54} - 80\,736u^{55}$$
$$+ 19\,674u^{56} + 207\,928u^{57} - 11\,292u^{58} - 274\,752u^{59} + 143\,032u^{60}.$$

A.2. Generating functions for three-state Potts model on square lattice

$F_0 = \ln f_1$

$F_1 = u^4\{2(4, 4)\}$

$F_2 = u^8\{4(6, 4, 2) + 4(6, 4, 0, 2) + 4(7, 6, 1) + 4(7, 6, 0, 1) - 18(8, 8)\}$

$F_3 = u^{12}\{258\tfrac{2}{3}(12, 12) - 112(11, 10, 1) - 112(11, 10, 0, 1)$

$\qquad\qquad + 24(10, 8, 1, 1) - 84(10, 8, 2) - 84(10, 8, 0, 2) + 8(8, 5, 2, 0, 1)$

$\qquad\qquad + 8(8, 5, 0, 2, 0, 1) + 16(8, 5, 1, 1, 0, 1) + 4(8, 4, 4) + 4(8, 4, 0, 4)$

$\qquad\qquad + 8(8, 4, 2, 2) + 16(9, 6, 0, 3) + 16(9, 6, 1, 2) + 16(9, 6, 2, 1) + 16(9, 6, 3)\}$

$F_4 = u^{16}\{-4532(16, 16) + 2896(15, 14, 1) + 2896(15, 14, 0, 1)$

$\qquad\qquad + 1720(14, 12, 2) + 1720(14, 12, 0, 2) - 1200(14, 12, 1, 1)$

$\qquad\qquad - 796(13, 10, 3) - 724(13, 10, 2, 1) - 724(13, 10, 1, 2)$

$\qquad\qquad - 796(13, 10, 0, 3) - 170(12, 8, 4) + 144(12, 8, 3, 1) - 332(12, 8, 2, 2)$

$\qquad\qquad + 144(12, 8, 1, 3) - 170(12, 8, 0, 4) + 40(11, 7, 3, 0, 1)$

$\qquad\qquad + 80(11, 7, 2, 1, 0, 1) + 40(11, 7, 2, 1, 1) + 120(11, 7, 1, 2, 0, 1)$

$\qquad\qquad + 40(11, 7, 0, 3, 0, 1) + 32(11, 6, 5) + 32(11, 6, 4, 1)$

$\qquad\qquad + 64(11, 6, 3, 2) + 64(11, 6, 2, 3) + 32(11, 6, 1, 4) + 32(11, 6, 0, 5)$

$\qquad\qquad - 240(12, 9, 2, 0, 1) - 480(12, 9, 1, 1, 0, 1) - 240(12, 9, 0, 2, 0, 1)$

$\qquad\qquad + 16(10, 6, 2, 0, 2) + 32(10, 6, 1, 1, 1, 1) + 32(10, 6, 1, 1, 0, 2)$

$\qquad\qquad + 32(10, 6, 0, 2, 0, 2) + 16(10, 6, 2, 0, 0, 2) + 16(10, 5, 4, 0, 1)$

$\qquad\qquad + 16(10, 5, 2, 2, 1) + 16(10, 5, 2, 2, 0, 1) + 32(10, 5, 3, 1, 0, 1)$

$\qquad\qquad + 32(10, 5, 1, 3, 0, 1) + 16(10, 5, 0, 4, 0, 1) + 4(10, 4, 6)$

$\qquad\qquad + 12(10, 4, 4, 2) + 12(10, 4, 2, 4) + 4(10, 4, 0, 6) + 2(9, 4, 4, 0, 0, 0, 0, 1)$

$\qquad\qquad + 8(9, 4, 2, 2, 0, 0, 0, 0, 1) + 4(9, 4, 2, 2, 0, 0, 0, 0, 0, 1)$

$\qquad\qquad + 2(9, 4, 0, 4, 0, 0, 0, 0, 0, 1).$

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